Unit 7: FRACTIONS

7.1.- UNDERSTANDING FRACTIONS

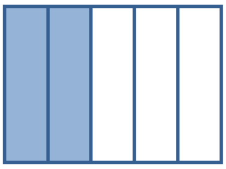
You use fractions when you ...

... cut a slice from a cake

Here is 1 out of the 6 pieces.

This is $\frac{1}{6}$ of the cake

... colour part of a shape blue



2 out of 5 parts are blue.

$\frac{2}{5}$ is blue

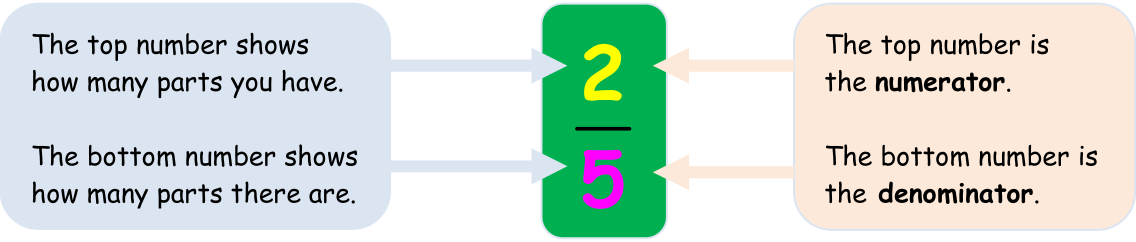
... tell the time

The minute hand is $\frac{1}{4}$ of the way round.

It's quarter past 6

A fraction describes part of a whole.

In a fraction:



The top number shows how many parts you have.

The bottom number shows how many parts there are.

$\frac{2}{5}$

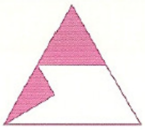
The top number is the **numerator**.

The bottom number is the **denominator**.

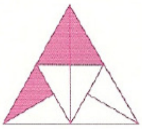
To use fractions the whole must be divided into equal-sized parts.

***Example 1*:**

What fraction of this shape is shaded?



First divide the triangle into equal pieces.



There are 8 equal parts. 3 parts are shaded. $\frac{3}{8}$ of the shape is shaded.

***Example 2*:** Peter has £1 and spends 35p on chocolate. What fraction of his money has he spent?

Divide his money into-equal sized parts: £1 = 100p.

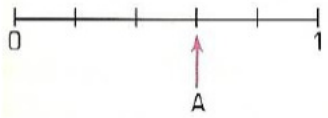
He has spent 35p. He has spent $\frac{35}{100}$ of his money.

Notice that the units have to be the same before you can compare them using fractions.

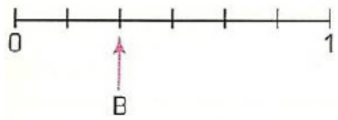
Exercise 1:

Use fractions to label the readings marked on each of these number lines:

a)



b)



Exercise 2:

What fraction of:

a) £1 is 20p
b) £10 is £3
c) 60 mm is 20 mm
d) 1 kg is 400 g
e) 60 mm is 3 cm
f) 250 cm is 2 m
g) 1 km is 150 m
h) 1 year is 72 hours

Exercise 3:

Design a flag with 4 colours. Each colour must be a different fraction of the flag. Describe what fraction of the flag is made from each colour.

7.2.- EQUIVALENT FRACTIONS

You can write the same fraction in different ways:

$\frac{1}{2}$ is shaded

$\frac{2}{4}$ is shaded

$\frac{3}{6}$ is shaded

The fractions $\frac{1}{2}$, $\frac{2}{4}$ and $\frac{3}{6}$ are all equivalent. You can write $\frac{1}{2}=\frac{2}{4}=\frac{3}{6}$.

You can write a fraction in its **simplest form** by **cancelling**.
You divide the numerator and the denominator by the highest common factor.

Examples:

a) $$\frac{15}{30}\ \begin{matrix}\xrightarrow{\div 15}\\ =\\ \xrightarrow[\div 15]{}\end{matrix}\ \frac{1}{2}$$

b) $$\frac{50}{120}\ \begin{matrix}\xrightarrow{\div 10}\\ =\\ \xrightarrow[\div 10]{}\end{matrix}\ \frac{5}{12}$$

You can find equivalent fractions by multiplying the numerator and denominator by the same number.

Examples:

a) $$\frac{2}{5}\ \begin{matrix}\xrightarrow{\times 6}\\ =\\ \xrightarrow[\times 6]{}\end{matrix}\ \frac{12}{30}$$

b) $$\frac{3}{4}\ \begin{matrix}\xrightarrow{\times 25}\\ =\\ \xrightarrow[\times 25]{}\end{matrix}\ \frac{75}{100}$$

Look at the previous

equivalent fractions:

$$\frac{2}{5}=\frac{2\times6}{5\times6} \qquad \boxed{2\times5\times6}=\boxed{5\times2\times6}$$

Observe that if you cross-multiply the numerators and denominators, you get the same number.

So you can test if two fractions are equivalent by cross-multiplying their numerators and denominators. This is also called **taking the cross-product**.

$$\boxed{\frac{a}{b}=\frac{c}{d}\Leftrightarrow a\cdot d=b\cdot c}$$

Example: Are $\frac{4}{6}$ and $\frac{6}{9}$ equivalent fractions?

$$4\times9=6\times6\ \Rightarrow\ \frac{4}{6}=\frac{6}{9}$$

The relationship between terms of equivalent fraction allows you to get each one of the terms if you know the others.

Example: $\frac{3}{4}=\frac{9}{x}\ \Rightarrow\ 3\cdot x=36\ \Rightarrow\ x=\frac{36}{3}=12$

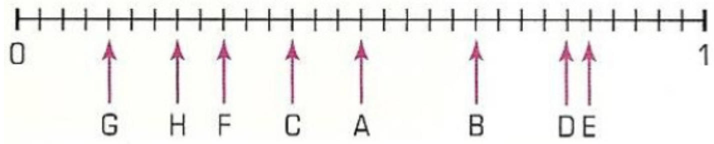
Exercise 4:

Cancel down each of these fractions to their simplest form by dividing the top and bottom number by a common factor.

$$\frac{7}{63},\ \frac{24}{60},\ \frac{5}{40},\ \frac{16}{36},\ \frac{25}{100},\ \frac{28}{42}$$

Exercise 5:

This number line has been split into 30 equal parts, so that each part is $\frac{1}{30}$.



Match each of these fractions to the letters indicated on the number line.

$$\boxed{\begin{array}{cccc}\frac{1}{2} & \frac{2}{3} & \frac{2}{5} & \frac{2}{15}\\ \frac{5}{6} & \frac{4}{5} & \frac{3}{10} & \frac{7}{30}\end{array}}$$

Exercise 6:

Calculate the value of x in each fraction.

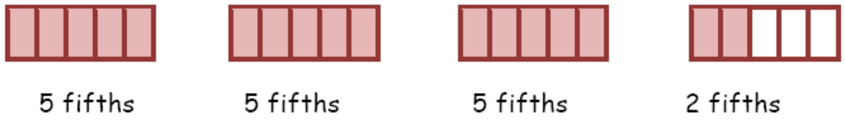
a) $\frac{2}{5}=\frac{x}{30}$ b) $\frac{4}{25}=\frac{16}{x}$ c) $\frac{x}{5}=\frac{6}{15}$ d) $\frac{6}{x}=\frac{30}{100}$

7.3.- MIXED NUMBERS. PROPER AND IMPROPER FRACTIONS

You often need to use numbers bigger than 1 which include fractions.

Numbers with a whole number part and a fraction are called **mixed numbers**.

Example: $3\frac{2}{5}$ means 3 whole numbers and 2 fifths.



There are
17 fifths in total.

So $3\frac{2}{5}=\frac{17}{5}$

A single fraction that is greater than 1, like $\frac{17}{5}$, is called an **improper fraction**.

A single fraction that is less than 1, like $\frac{2}{5}$, is called a **proper fraction**.

How to convert mixed numbers to improper fractions?

Multiply the whole number by the denominator and then add the numerator of the proper fraction. The number you get is the numerator of the improper fraction.

The denominator is the same as the mixed number has.

Example: $$2\frac{5}{7} \xrightarrow[7\times 2+5=14+5=19]{} 2\frac{5}{7}=\frac{19}{7}$$

How to convert improper fraction to mixed numbers?

$$\frac{12}{7}=\frac{7}{7}+\frac{5}{7}=1+\frac{5}{7}\Rightarrow\frac{12}{7}=1\frac{5}{7} \qquad \frac{7}{3}=\frac{3}{3}+\frac{3}{3}+\frac{1}{3}=2+\frac{1}{3}\Rightarrow\frac{7}{3}=2\frac{1}{3}$$

A quick way to convert $\frac{7}{3}$ is to divide by 3.

$$\begin{array}{l|l} 7 & 3 \\ \hline 1 & 2 \end{array} \Longrightarrow \boxed{\frac{7}{3}=2+\frac{1}{3}}$$

D= dividend
d= divisor
q = quotient
r = remainder

$$\begin{array}{l|l} D & d \\ \cline{2-2} r & q \end{array} \quad \Longrightarrow \quad \boxed{\frac{D}{d} = q + \frac{r}{d}}$$

Exercise 7:

Convert these mixed numbers to improper fractions.

a) $8\frac{2}{3}$ b) $2\frac{4}{5}$ c) $1\frac{3}{4}$ d) $3\frac{5}{8}$ e) $5\frac{7}{12}$

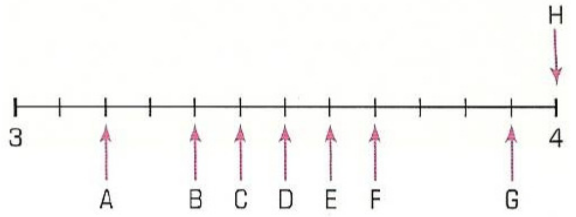
Exercise 8:

Convert each of these improper fractions to mixed numbers.

a) $\frac{43}{3}$ b) $\frac{49}{10}$ c) $\frac{27}{7}$ d) $\frac{15}{4}$ e) $\frac{32}{5}$

Exercise 9:

This number line has been split into 12 equal parts, so that each part is $\frac{1}{12}$.



Match each of these fractions to the letters indicated on the number line.

$3\frac{7}{12}$ $\frac{19}{6}$ $\frac{94}{24}$ $\frac{16}{4}$

$3\frac{2}{3}$ $\frac{56}{16}$ $\frac{200}{60}$ $3\frac{20}{48}$

7.4.- CONVERTING FRACTIONS TO DECIMALS

To convert a fraction to a decimal divide the numerator by the denominator.

Example: Write these fractions as a decimals.

a) $\frac{13}{8}$ b) $\frac{1}{7}$ c) $\frac{2}{15}$

a) $\frac{13}{8} = 1.625$

1.625 is a **terminating** decimal.

b) $\frac{1}{7} = 0.142857142857142...$

0.142857142... $= 0.\dot{1}4285\dot{7}$ is a **recurring** decimal. The dots show the recurring group of digits.

c) $\frac{2}{15} = 0.13333333...$

0.1333333... $= 0.1\dot{3}$ is a **recurring** decimal. The dot over the 3 shows the recurring decimal.

To decide if a fraction will be a terminating or a recurring decimal, look at the denominator.

- If the only factors of the denominator are 2 and/or 5 or combinations of 2 and 5 then the fraction will be a **terminating decimal**.
- If the denominator has any factors other than 2 and/or 5 then the fraction will be a **recurring decimal**.

Exercise 10:

Convert these fractions to decimals.

a) $\frac{3}{7}$ b) $\frac{5}{9}$ c) $\frac{4}{25}$ d) $\frac{17}{80}$ e) $\frac{31}{13}$

Exercise 11:

Without doing the division, state whether each of these fractions will give a recurring decimal or a terminating decimal.

a) $\frac{1}{25}$ b) $\frac{4}{11}$ c) $\frac{3}{20}$ d) $\frac{3}{126}$ e) $\frac{11}{128}$

7.5.- COMPARING FRACTIONS

Sometimes we need to compare two fractions to discover which is larger or smaller. There are two easy ways to compare fractions: using decimals or using the same denominator.

The decimal method of comparing fractions

Just convert each fraction to decimals, and then compare the decimals.

Example: Which is bigger, $\frac{3}{8}$ or $\frac{5}{12}$?

$\frac{3}{8} = 0.375$ $\frac{5}{12} = 0.4166...$ $\Longrightarrow$ $\frac{5}{12}$ is bigger

The same denominator method

If two fractions have the same denominator then they are easy to compare:

$\frac{2}{4}$

$\frac{3}{4}$

$\frac{2}{4} < \frac{3}{4}$ (because 2 is less than 3)

To compare two or more fractions with different denominators:

- Find the common denominator.
- Work out the equivalent fractions.
- Write the fractions in ascending or descending order.

Example: Write these in ascending order. $\frac{7}{8}$ $\frac{5}{6}$ $\frac{3}{4}$

The common denominator will be the least common multiple (LCM) of the denominators.

$$\text{LCM}(8, 6, 4) = 24$$

$$\left.\begin{array}{ll}\frac{7}{8}=\frac{21}{24} & \text{Multiply numerator and denominator by 3}\\[6pt] \frac{5}{6}=\frac{20}{24} & \text{Multiply numerator and denominator by 4}\\[6pt] \frac{3}{4}=\frac{18}{24} & \text{Multiply numerator and denominator by 6}\end{array}\right\} \quad \frac{18}{24}<\frac{20}{24}<\frac{21}{24}$$

In ascending order the fractions are: $\frac{3}{4}, \frac{5}{6}, \frac{7}{8}$

Exercise 12:

Adam cut his pie into 8 pieces and ate 5. Rachel cut her pie into 6 pieces and ate 3. Who has eaten a larger portion?

Exercise 13:

Put these fractions in order from smallest to largest: $\frac{4}{7}, \frac{5}{8}$ and $\frac{9}{14}$.

Exercise 14:

For each pair of fractions insert the appropriate sign Choose from: <, > or =

a) $4\frac{1}{3}$ $\frac{15}{3}$ b) $\frac{37}{5}$ $5\frac{2}{5}$ c) $3\frac{2}{3}$ $\frac{44}{12}$ d) $\frac{14}{3}$ $4\frac{3}{4}$ e) $7\frac{1}{2}$ $\frac{23}{5}$

7.6.- ADDING AND SUBTRACTING FRACTIONS

It is easy to add or subtract fractions when they have the same denominator.

$$\frac{3}{8}+\frac{1}{8}=\frac{4}{8}$$

You can add or subtract fractions with different denominator by first writing them as **equivalent fractions** with the same denominator.

Example: Calculate a) $\frac{3}{5}+\frac{1}{3}$ b) $1\frac{3}{4}-\frac{5}{7}$ c) $1\frac{7}{10}+2\frac{3}{5}$

a) $\frac{3}{5}+\frac{1}{3}=\frac{9}{15}+\frac{5}{15}=\frac{14}{15}$

b) $1\frac{3}{4}-\frac{5}{7}=\frac{7}{4}-\frac{5}{7}=\frac{49}{28}-\frac{20}{28}=\frac{29}{28}=1\frac{1}{28}$

c) $1\frac{7}{10}+2\frac{3}{5}=\frac{17}{10}+\frac{13}{5}=\frac{17}{10}+\frac{26}{10}=\frac{43}{10}=4\frac{3}{10}$

An alternative method is to write:

$1+\frac{7}{10}+2+\frac{3}{5}=3+\frac{7}{10}+\frac{3}{5}=\ldots$

Exercise 15:

Peter walked $3\frac{2}{3}$ miles before lunch and then a further $2\frac{1}{4}$ miles after lunch. How far did he walk altogether?

Exercise 16:

A bag weighs $2\frac{3}{16}$ lb when it is full. When empty the bag weighs $\frac{3}{8}$ lb. What is the weight of the content of the bag?

Exercise 17:

Henry and Paula are eating pistachios. Henry was a full bag weighing $1\frac{3}{16}$ kg. Paula has a bag that weighs $\frac{4}{5}$ kg. What is the total mass of their two bags of pistachios?

Exercise 18:

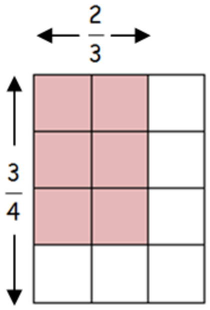
Simon spent $\frac{2}{3}$ of his wages on a mobile phone. He spent $\frac{1}{5}$ of his wages on a trip to the theatre. Work out the fraction of his wages that he had left.

7.7.- MULTIPLYING AND DIVIDING FRACTIONS

The diagram shows the multiplication

$$\frac{2}{3}\cdot\frac{3}{4}=\frac{6}{12}=\frac{1}{2}$$

You get the same result if you multiply the numerators together and multiply the denominators together.



To **multiply** fractions, multiply the numerators and then multiply the denominators, then cancel any common factors.

Examples: a) $\frac{2}{3}\cdot\frac{4}{5}=\frac{2\cdot 4}{3\cdot 5}=\frac{8}{15}$ b) $\frac{4}{9}\cdot\frac{3}{5}=\frac{4\cdot 3}{9\cdot 5}=\frac{12}{45}=\frac{4}{15}$

A unit fraction has a numerator of 1. For example, $\frac{1}{5}$.

The **multiplicative inverse** of an integer is its **reciprocal**.

For example, the reciprocal of 3 is $\frac{1}{3}$.

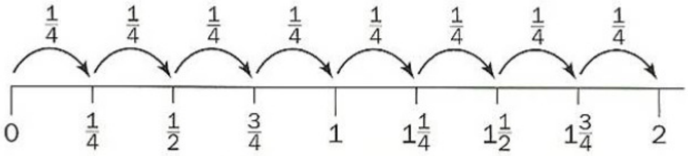
The **multiplicative inverse** of a fraction is the original fraction 'turned upside down'

The inverse of $\frac{3}{5}$ is $\frac{5}{3}$.

Multiplying by a unit fraction is the same as dividing by its denominator. For example, multiplying by $\frac{1}{5}$ is the same as dividing by 5.

$$\boxed{10\cdot\frac{1}{5}=\frac{10}{5}=2 \;\Leftrightarrow\; 10:5=2}$$

Dividing by a unit fraction is the same as multiplying by its denominator.



$$\boxed{2:\frac{1}{4}=8 \;\Leftrightarrow\; 2\cdot 4=8}$$

To divide by a fraction, multiply by its multiplicative inverse.

Example: $\frac{1}{2}:\frac{1}{6}=\frac{1}{2}\cdot\frac{6}{1}=\frac{1\cdot 6}{2\cdot 1}=\frac{6}{2}=3$

Does it make sense?

Does $\frac{1}{2}:\frac{1}{6}$ really equal 3?

You can change a question like 'What is 20 divided by 5?' into 'How many 5s fit into 20?'

In the same way our fraction question can become:

$$\frac{1}{2}:\frac{1}{6} \qquad \Longrightarrow \qquad \text{How many } \frac{1}{6} \text{ in } \frac{1}{2}?$$

Now look at the pizzas below... how many $\frac{1}{6}$ slices fit into a $\frac{1}{2}$ slice?

How many  in  ? Answer: 3

So now you cans see that $\frac{1}{2}:\frac{1}{6}=3$ really does make sense!

To **divide** two fractions, multiply the first by the multiplicative inverse of the second. This is the same as cross-multiplying their numerators and denominators.

$$\frac{7}{8}:\frac{5}{6}=\frac{7}{8}\cdot\frac{6}{5}=\frac{7\cdot 6}{8\cdot 5}=\frac{42}{40}=\frac{21}{20}=1\frac{1}{20}$$

You can use the relationship between multiplication and division.

$$\begin{tikzcd} 4 \arrow[r, bend left, ":\frac{2}{3}"] & 6 \arrow[l, bend left, "\times\frac{2}{3}"] \end{tikzcd}$$

Dividing by $\frac{2}{3}$ is the same as multiplying by $\frac{3}{2}$.

Multiplying by $\frac{2}{3}$ is the same as dividing by $\frac{3}{2}$.

$$\begin{tikzcd} 4 \arrow[r, bend left, "\times\frac{3}{2}"] & 6 \arrow[l, bend left, ":\frac{3}{2}"] \end{tikzcd}$$

Exercise 19:

Rewrite each of these divisions as multiplications.

a) 8 : 5　　b) 6 : 4　　c) 12 : 7　　d) 17 : 3　　e) 7:2

Exercise 20:

Calculate these, giving your answers in their simplest form.

a) $\frac{1}{4}\cdot\frac{3}{5}$　　b) $6\cdot\frac{7}{4}$　　c) $\frac{8}{35}\cdot\frac{7}{24}$　　d) $\frac{4}{5}\cdot\frac{3}{13}$　　e) $\frac{7}{8}\cdot 12$

Exercise 21:

Copy and complete these sentences.

a) Dividing a number by 4 is the same as multiplying the number by _______.

b) Multiplying a number by $\frac{1}{2}$ is the same as dividing the number by _______.

c) Dividing a number by $\frac{1}{3}$ is the same as multiplying the number by _______.

Exercise 22:

Calculate these, giving your answers in their simplest form.

a) $\frac{5}{8}:\frac{7}{9}$　　b) $11:\frac{2}{3}$　　c) $\frac{1}{12}:\frac{3}{8}$　　d) $\frac{4}{5}:\frac{7}{8}$　　e) $\frac{3}{16}:9$

Exercise 23:

One ibuprofen tablet contains $\frac{1}{5}$ of a gram of active ingredient. How many milligrams are there in three tablets?

Exercise 24:

A sheet of paper has dimensions $8\frac{1}{2}$ inches by 11 inches. How many square inches is the paper?

Exercise 25:

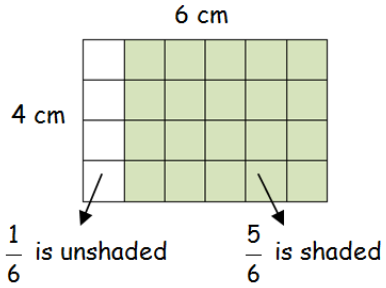
A track is $\frac{1}{3}$ of a mile long. How many times does John have to run around the track if he wants to run 6 miles?

Exercise 26:

A paint pot can hold $2\frac{3}{4}$ litres of paint. Hector buys $11\frac{1}{5}$ litres of emulsion paint. Hoy many times can Hector fill the paint pot with emulsion paint?

7.8.- FINDING FRACTIONS OF QUANTITIES

This rectangle has an area of 24 cm^2.



The unshaded area of the rectangle is $\frac{1}{6}$ of 24 cm^2.

The shaded area of the rectangle is $\frac{5}{6}$ of 24 cm^2.

To find **one sixth** of a number we **divide the number by six**.

$$\frac{1}{6} \text{ of } 24 \text{ cm}^2 = 24 \text{ cm}^2 : 6 = 4 \text{ cm}^2$$

Then, to find **five sixths** of a number, we first find one sixth of that number and then **multiply this by five**.

$$\frac{1}{6} \text{ of } 24 \text{ cm}^2 = 24 \text{ cm}^2 : 6 = 4 \text{ cm}^2; \qquad 5 \cdot 4 \text{ cm}^2 = 20 \text{ cm}^2$$

Dividing 24 by 6 and multiplying the number you get by 5 is the same as multiplying 24 by $\frac{5}{6}$.

Therefore, <u>you find fractions of a quantity by multiplying</u>.

For example, $\frac{2}{3}$ of 5 $= \frac{2}{3} \cdot 5 = \frac{2 \cdot 5}{3} = \frac{10}{3}$

You can extend this method **to finding a fraction of a fraction of a quantity**.

Example:

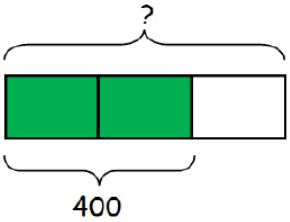
Tom has £42. He spends $\frac{1}{3}$ of it on Monday. On Tuesday he spends $\frac{3}{4}$ of the remainder. How much does he spend on Tuesday?

Tom spends $\frac{1}{3} \cdot$ £42 on Monday, so he has $\frac{2}{3} \cdot$ £42 on Tuesday.

$$\frac{3}{4} \text{ of } \frac{2}{3} \cdot 42 \text{ is } \frac{3}{4} \cdot \frac{2}{3} \cdot 42 = £21$$

Now, we move on the "inverse problem":

Example: The two thirds of a quantity are 400. What is the quantity?



If the two thirds of a quantity are 400, one third of this quantity is $400 : 2 = 200$.

$\frac{1}{3}$ of a quantity is 200 $\Rightarrow$ the whole quantity is $200 \cdot 3 = 600$

Dividing 400 by 2 and multiplying the number you get by 3 is the same as **multiplying 400 by $\frac{3}{2}$** (the multiplicative inverse of $\frac{2}{3}$).

More examples:

$$\frac{4}{5} \text{ of } C = 240 \;\Rightarrow\; C = 240 \cdot \frac{5}{4} = 300$$

$$\frac{3}{7} \text{ of } C = 60 \;\Rightarrow\; C = 60 \cdot \frac{7}{3} = 140$$

To sum up:

The fraction $\frac{a}{b}$ of C is equal to $\frac{a}{b} \cdot C$

If the fraction $\frac{a}{b}$ of C is equal to P, then C is equal to $P \cdot \frac{b}{a}$

Exercise 27:

Calculate the amount of liquid in these containers.

a) A 40 litre barrel that is $\frac{3}{8}$ full.

b) A 240 cl jar that is $\frac{3}{4}$ full.

c) A 120 cl glass that is $\frac{2}{5}$ empty.

d) A 750 ml litre bottle that is $\frac{2}{3}$ empty.

Exercise 28:

$\frac{3}{5}$ of a group of children were girls. If there were 24 girls, how many children were there in the group?

Exercise 29:

A store sold $\frac{5}{8}$ of its 48 rose plants at the full price of \$24 and sold the rest for $\frac{2}{3}$ of the full price. How much money did they take in for the rose plants?

Exercise 30:

An empty swimming pool is to be filled with water. It takes 12 hours to fill the pool, and the full pool contains 98 m^3 of water. How much water will the pool contain after 5 hours? Show your work.

Exercise 31 :

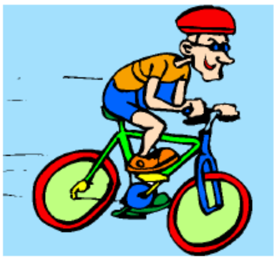
Calculate these times using fractions, and then convert each of your answers into hours and minutes.

a) $\frac{7}{12}$ of 9 hours b) $\frac{5}{8}$ of 22 hours c) $\frac{7}{10}$ of 1 day

Exercise 32:

Pete the Painter can paint $\frac{5}{6}$ of a room in 90 minutes. How many minutes will take Pete to paint the entire room?

Exercise 33:



A cyclist has covered $\frac{3}{5}$ of the total cycle race. If there are 30 kilometres left to finish the race, how long is the total race?

Exercise 34:

$\frac{5}{7}$ of a group of students are boys. There are 18 more boys than girls. How many students are there altogether?

Exercise 35:

Karen promised Sara that she would give half of her remaining money to her after giving $\frac{7}{8}$ of it to her favourite charity. If Karen gave $672 to her favourite charity, how much money will Sara receive?

Exercise 36:

Gary works at Ghosts and Goblins, a Halloween store. He puts orange, black, and purple Halloween candles in boxes. After he fills a box, $\frac{1}{4}$ is orange, $\frac{1}{6}$ is black, and 21 are purple. How many candles does Gary put in each box?

Exercise 37:

Benjamin found a bag of money. He decided to give $\frac{1}{2}$ of the money to his brother, $\frac{1}{4}$ of the money ho his sister, $\frac{1}{8}$ of the money to his mother, and $\frac{1}{16}$ of the money to his dad. The amount of money that was left was $1.25. How much money was in the bag when Benjamin found it?

Exercise 38:

Marcus had $450. He spent $\frac{2}{5}$ of it on a DVD player. He then spent $\frac{1}{3}$ of the remaining money on a jacket. How much money did Marcus have left?

Exercise 39:

Chef Pillsbury purchased some eggs. He used $\frac{1}{2}$ of them to make some pies. He used $\frac{1}{4}$ of the remaining eggs to make a cake. He then had 15 eggs left. How many eggs had Chef Pillsbury purchased?

Exercise 40:

Kara read 20 pages of her new book on Saturday. She read $\frac{1}{4}$ of the remaining pages on Sunday. She still has 36 pages to read. How many pages are in Kara's book?