Unit 8: LINEAR AND QUADRATIC FUNCTIONS

This unit will show you how to:

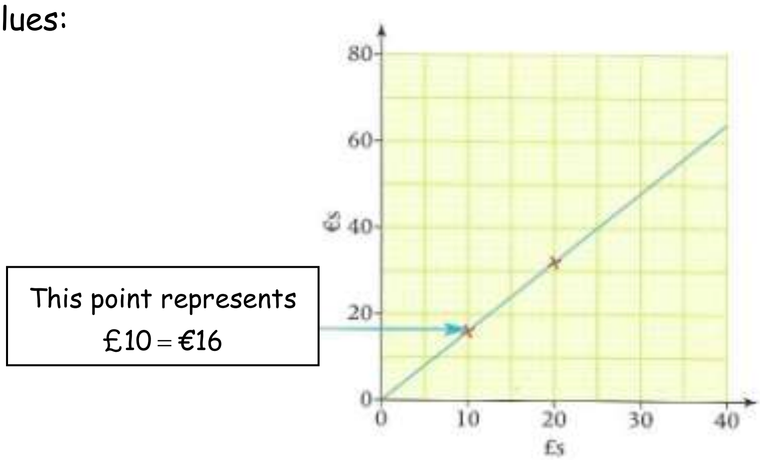
- Recognize that equations of the form $y = mx + n$ have straight line graphs
- Plot straight line graphs given a linear equation
- Recognize and understand the form of equations corresponding to horizontal, vertical and diagonal line graphs
- Find the slope and y-axis intercept of straight line graphs
- Form linear functions, using the corresponding graphs to solve real-life problems
- Plot parabolas given a quadratic equation and used them to solve real-life problems

Keywords	
Linear	Slope (gradient)
Diagonal	Raise
Horizontal	Run
Intersect	x-intercept
Vertical	y-intercept
Coefficient	parabola
Constant	vertex

8.1.- STRAIGHT LINE GRAPHS

Example 1: Davina buys some euros for a trip to France. The exchange rate is £1=€1.60. She draws a conversion graph to help her convert prices. First she draws up a table with some values:

Pounds (x)	Euros (y)
0	0
10	16
20	32
30	48
40	64



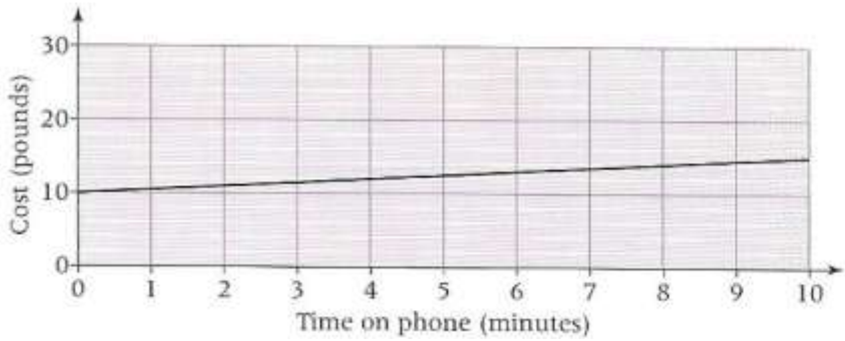
The function that converts pounds in euros is: $y = 1.60x$.

Example 2: Suppose you have a mobile phone. You pay £10 line rental each month, then 50 p for every minute you spend making calls.

Time on calls in minutes (x)	0	1	2	3	4
Price of calls in pounds	0	0.50	1.00	1.50	2.00
Line Rental (£)	10	10	10	10	10
Total cost in pounds (y)	10	10.50	11	11.50	12

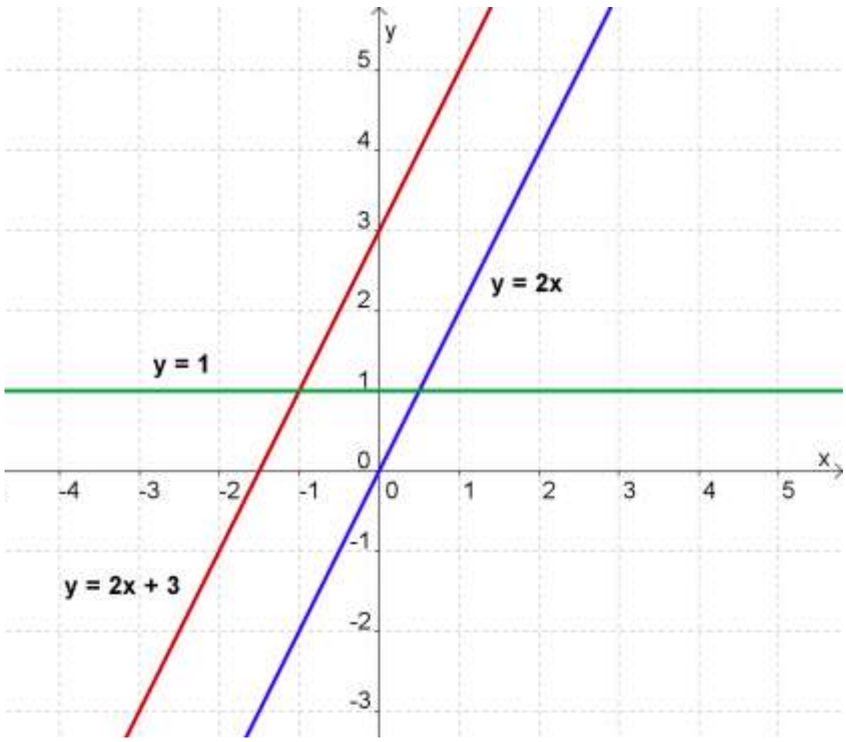
To get the total cost, you add £10 (the line rental) to the call cost. The call cost is the number of minutes on the phone multiplied by 50. Hence,

$$\text{Total cost} = 0.50 \times \text{time on phone} + 10 \;\Rightarrow\; y = 0.5x + 10$$



Any function of the form $\mathbf{y = mx + c}$ is called a **linear function**. Its graph is a **straight line**.

Examples: $y = 2x + 3$, $y = 2x$, $y = 1$

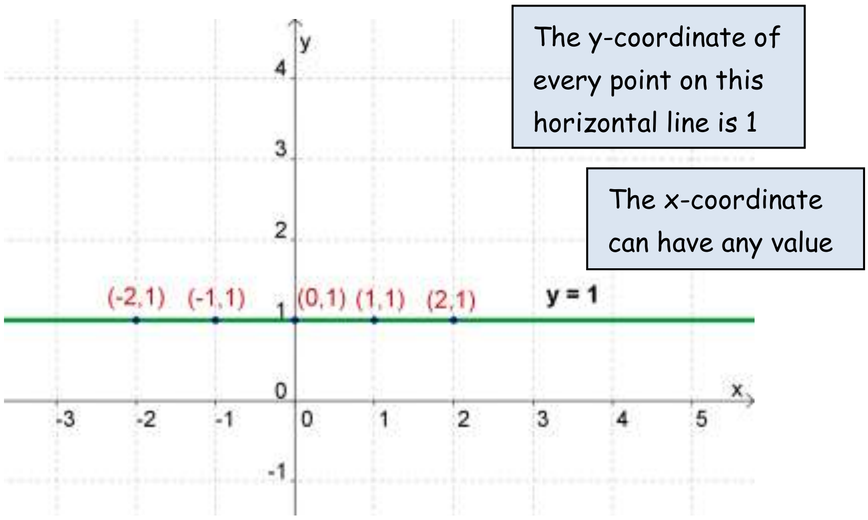


- Linear functions in which $c = 0$, that is, $\mathbf{y = mx}$, are called **proportionality functions**. The variable "y" is directly proportional to "x". The constant ratio, $m = \frac{y}{x}$, is called **proportionality constant** (or constant of proportionality). Their graphs pass through the point $(0, 0)$.

  Examples: $y = 1.60x$, $y = 5x$

- Linear functions in which $m = 0$, that is, $\mathbf{y = c}$, are called **constant functions**. Their graphs are horizontal lines.

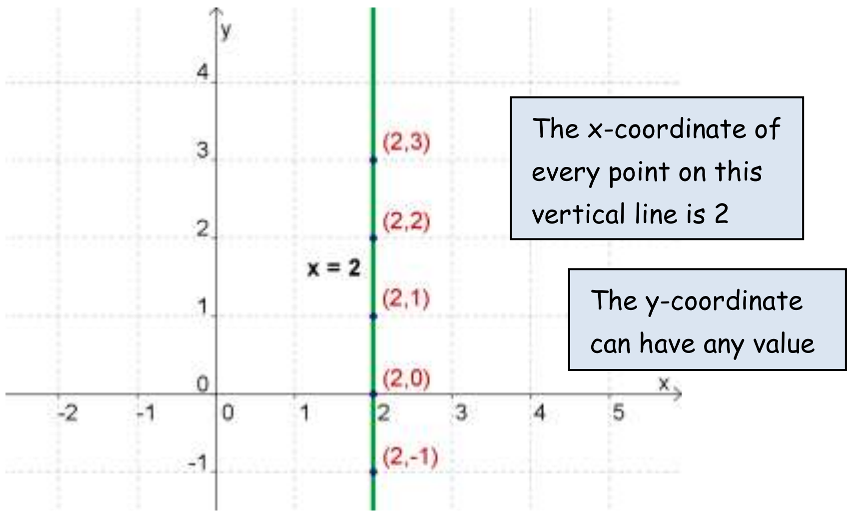
  Example: $y = 1$



Up to now, you have seen **diagonal** and **horizontal lines**.

- **Diagonal lines** have equations of the form $\mathbf{y = mx + c}$, where $m \neq 0$.
- **Horizontal lines** have equations of the form $\mathbf{y = c}$.

But there is another kind of straight lines: **vertical lines**.



Notice that **vertical lines are not functions**, because there are infinite values of "y" corresponding the same value of "x".

Vertical lines have equations of the form $\mathbf{x = c}$.

Exercise 1

Which of these equations have straight line graphs?

$y = 2x + 3$	$y = 7 - 3x$	$y = x^2 + 1$	$y = 5x$
$y = 7$	$y = 2x - x^3$	$2x + 7y = 8$	$x = -2$

Exercise 2

Copy and complete the table, deciding if each equation is that of a horizontal, vertical or diagonal line, or none of these.

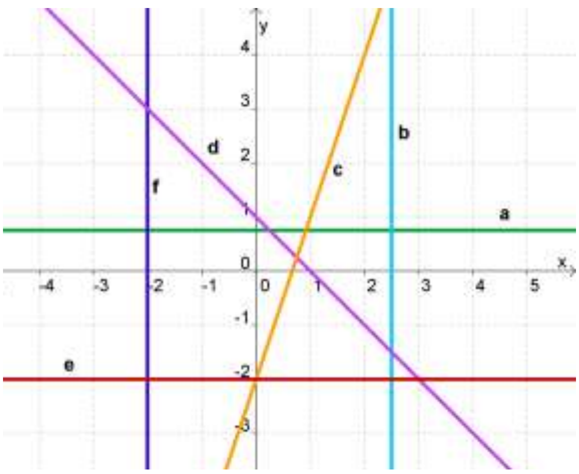
$x = 9$	$y = 2x - 1$	$x = -0.5$	$y = x^2 + x$
$xy = 1$	$y = 0$	$y = -x + 3$	$y = 7$

Horizontal	Vertical	Diagonal	None of these

Exercise 3

Match each line with its equation.

$y = -2$	$y = -x + 1$	$x = 2.5$	$x = -2$	$y = 0.75$	$y = 3x - 2$



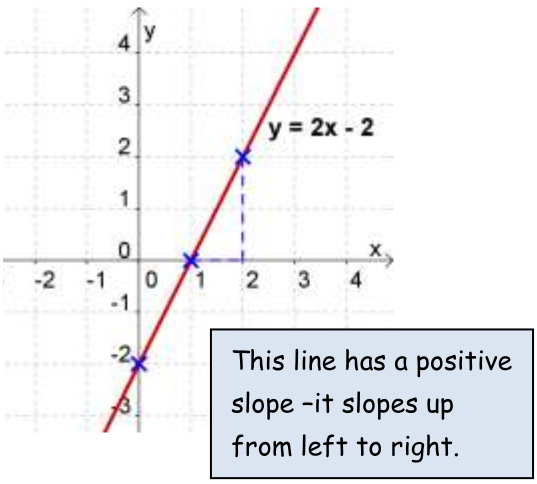
Exercise 4

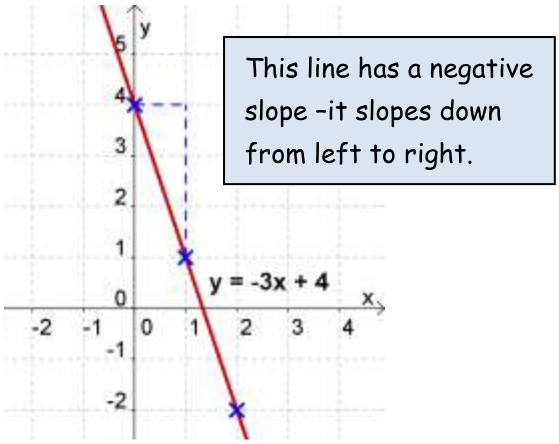
a) Give the equations of four lines which, when plotted, form the sides of a rectangle.
b) Repeat part a) for a square.
c) Repeat part a) for an isosceles right-angled triangle.

8.2.- SLOPE AND INTERCEPTS

The **slope (or gradient)** of a straight line tells you how steep it is.

To work out the slope find how many units the line **rises** for each unit it **runs** across the page.





For the line $y = 2x - 2$,

$$\textbf{slope} = \frac{\textbf{rise}}{\textbf{run}} = \frac{2}{1} = 2$$

For the line $y = -3x + 4$,

$$\textbf{slope} = \frac{\textbf{rise}}{\textbf{run}} = \frac{-3}{1} = -3$$

The slope is the **coefficient** of x (the number of xs) in the equation of the line.

- Straight lines with **positive slope** are **increasing functions**.
- Straight lines with **negative slope** are **decreasing functions**.
- The **slope** of **constant functions** is **0**.

The **y-intercept** is the distance from the origin to where the line cuts the y-axis.

The line $y = 2x - 2$ cuts the y-axis at $(0, -2)$. The y-intercept is -2.

The line $y = -3x + 4$ cuts the y-axis at $(0, 4)$. The y-intercept is 4.

The y-intercept is the **constant** term (the number) in the equation of the line.

Example: Find the slope and the y-intercept of the lines

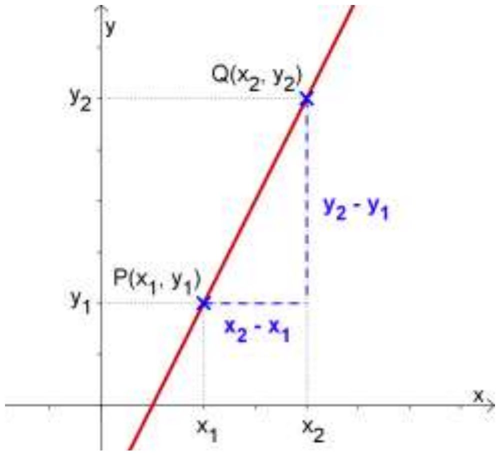
a) $y=-5x+2$ b) $y=-2$ c) $y=7x$ d) $3x+2y=12$

a) $\text{slope}=-5$, $\text{y-intercept}=2$
b) $\text{slope}=0$, $\text{y-intercept}=-2$
c) $\text{slope}=7$, $\text{y-intercept}=0$
d) The equation is not in the form $y=mx+c$, so rearrange it first:

$$3x+2y=12 \Rightarrow 2y=-3x+12 \Rightarrow y=\frac{-3x+12}{2} \Rightarrow y=-\frac{3}{2}x+6$$

Now you can see that the slope is $-\frac{3}{2}$ and the y-intercept is 6.

How to calculate the slope of a line if you know two points of it?



$$\text{slope}=\frac{\text{rise}}{\text{run}}=\frac{\text{change in } y}{\text{change in } x}$$

The slope of the line joining $P(x_1, y_1)$ and $Q(x_2, y_2)$ is:

$$\textbf{slope}=\frac{y_2-y_1}{x_2-x_1}$$

The slope is denoted by **m**.

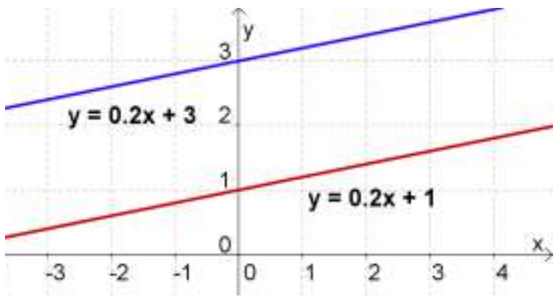
Example: The slope of the line joining $P(-3, 2)$ and $Q(5, 1)$ is:

$$m=\frac{1-2}{5-(-3)}=\frac{-1}{8}$$

Since the slope of a straight line tells you how steep it is, **parallel** lines will have the same slope.

Example: Find the equation of a line parallel to $y=0.2x+3$.

The line $y=0.2x+3$ has slope 0.2.
A line that is parallel to it will have the same slope.
So, $y=0.2x+1$ is parallel to the line $y=0.2x+3$.



8.3.- THE EQUATION $y = mx + c$

You can write the equation of any straight line in the form $y = mx + c$

For example: $x + y = 5 \Rightarrow y = -x + 5$

$4x - y = 6 \Rightarrow y = 4x - 6$

The **slope** of the line $y = mx + c$ is **m**.

The **y-axis intercept** of the line $y = mx + c$ is **c**.

m is the **coefficient** of x

c is the **constant** (number)

Example: What are the slope and intercept of each of these lines?

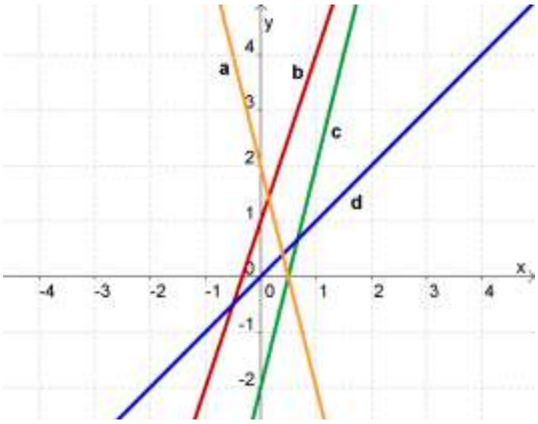
a) $y = 2x + 5$ b) $y = 1 - 3x$

a) $y = 2x + 5 \Rightarrow m = 2, \quad c = 5$ b) $y = 1 - 3x \Rightarrow m = -3, \quad c = 1$

Exercise 5

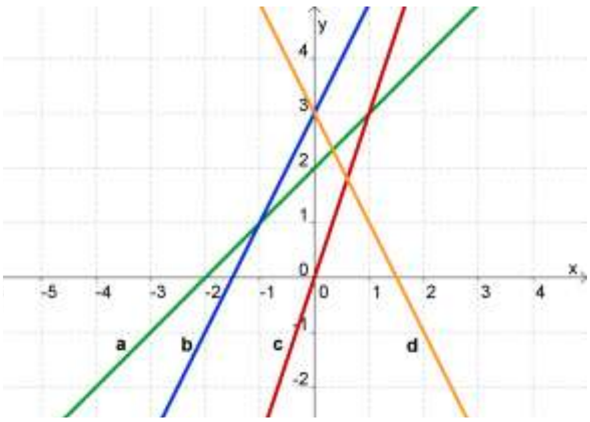
Match each line with its equation.

$y = 4x - 2$ $y = 3x + 1$ $y = x$ $y = 2 - 4x$



Exercise 6

Find the slope and the intercept of these graphs. Write the equation of each line.



Exercise 7

Write the equation of a straight line that is parallel to $y = 7 - 2x$ and cuts the y-axis at (0, 3).

8.4.- FINDING THE EQUATION OF A STRAIGHT LINE GRAPH

If you know the **slope** of a line and the y-axis **intercept** you can write the equation of the line.

Example: What is the equation of a line with slope 9 passing through (0, 5)?

Slope $= 9$ and intercept $= 5$, so the equation of the line is $y = 9x + 5$.

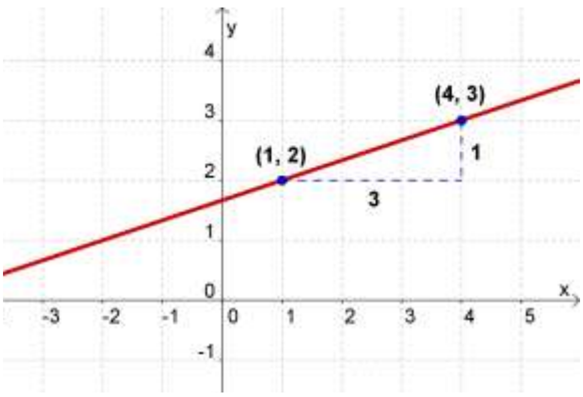
If you know the **slope** and a **point** on the line you can find the equation of the line.

Example: What is the equation of a line with slope 8 that passes through the point (2, 7)?

Slope $= 9$, so the equation of the line is $y = 8x + c$.
The line goes through (2, 7) so,
$7 = 8 \cdot 2 + c$ (Put $x = 2$ and $y = 7$ in the equation $y = 8x + c$)
$7 = 16 + c \Rightarrow c = -9$
The equation of the line is $y = 8x - 9$

If you know **two points** on a line you can find the equation of the line.

Example: Find the equation of the line joining (1, 2) and (4, 3).



$$\text{slope} = \frac{\text{rise}}{\text{run}} = \frac{3-2}{4-1} = \frac{1}{3}$$

The line goes through (1, 2) so substitute 1 for x and 2 for y in $y = \frac{1}{3}x + c$.

$$2 = \frac{1}{3} \cdot 1 + c \Rightarrow 2 = \frac{1}{3} + c \Rightarrow c = \frac{5}{3}$$

The equation is $y = \frac{1}{3}x + \frac{5}{3}$.

Exercise 8

Find the equations of the nine lines described in the table.

a) Slope of 7 and intercepts y-axis at (0,5)	**b)** Slope of 0.5 and passes through (0, 3)	**c)** Parallel to a line with slope 4 and passing through (3, 8)
d) Slope of 3 and passing through (4, 7)	**e)** Slope of -2 and cutting through (4, -3)	**f)** Parallel to $y = \frac{1}{4}x - 1$ and passing through (0, -2)
g) Passing through (0, 1) and (1, 5)	**h)** Passing through (0, 2) and (5, 7)	**i)** Passing through the midpoint of (1, 7) and (3, 13) with a slope of 8

Exercise 9

Where does the line $2y = 9x - 5$ cross

a) the y-axis   b) the x-axis   c) the line $4y = x + 24$?

8.4.- LINEAR GRAPHS IN REAL LIFE

You can use a graph to represent a real-life situation. Remember the examples 1 and 2 at the beginning of the unit.

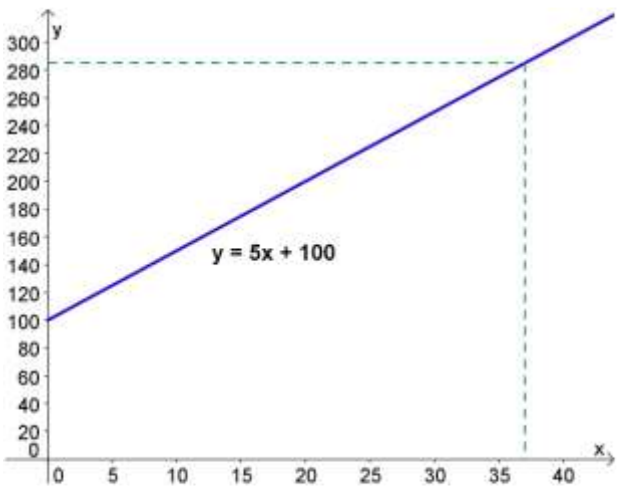
Let's see some more examples:

I) *Plot a graph to represent the total cost of hiring a party venue, if the owner charges £100 hire fee and £5 per guest. Use the graph to estimate the number of guests if the total bill is £285.*

If **x** is the **number of guests** and **y** is the **cost** (£s), then $y = 5x + 100$

Draw a horizontal line from £285 to the graph.
Draw a vertical line from the graph to the vertical axis.

For cost of £285, number of guests $= 37$.



You can also get the number of guests by using the analytical expression of the function: $y = 5x + 100$.

$$\text{If } y = 285 \Rightarrow 285 = 5x + 100 \Rightarrow 5x = 185 \Rightarrow x = 37$$

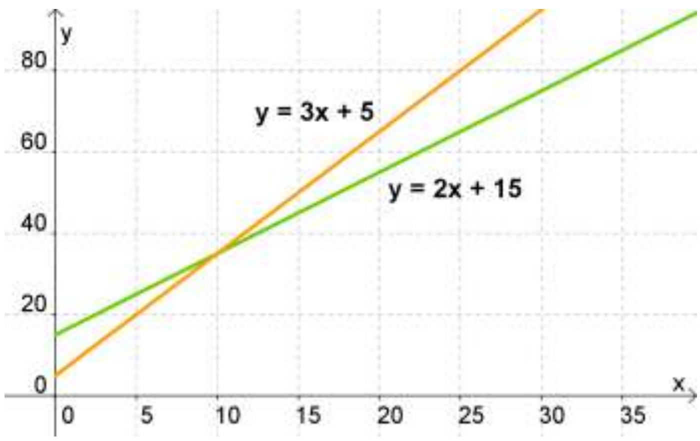
II) *Two competing electricity companies use these formulae to work out customers' bills.*

POWER UP!
$y = 3x + 5$

SPARKS ARE US!
$y = 2x + 15$

The number of units of electricity used is x. The price of the electricity is £y.

Using graphs, compare the pricing policies of the two companies and advise householders from which company they should buy their electricity.



The graphs show that, from 0 to 10 units of electricity used, the company POWER UP! is cheaper than the company SPARKS ARE US!. However, if the units used are more than 10, the cheapest company is SPARKS ARE US!.

Exercise 10

A 500-liter tank full of oil is being drained at the constant rate of 20 litres per minute.

a) Write a linear function "y" for the number of litres in the tank after "x" minutes (assuming that the drainage started at $x = 0$).
b) Find the y-intercept and interpret it.
c) How many litres are in the tank after 11 minutes and 45 seconds?

Exercise 11

Pauline and her family are going on holiday and exchange £400 spending money into euros (€) before they go.

At the bank, the exchange rate is £1 = €1.45.

a) Construct a graph that the family can take on holiday to convert any amount of their spending money from pounds to euros or vice versa.
b) Use the graph to find
   i. the cost, in euros, of a side trip which is advertised for £95
   ii. the cost, in pounds, of a meal in a restaurant that comes to €85.

Exercise 12

The melting point of the ice is 0°C (or 32°F) and the boiling point of the water is 100°C (or 212°F).

a) Write a linear function that converts any temperature from Celsius degrees (°C) to Fahrenheit degrees (°F).
b) Draw the graph of this function.
c) Complete the table below:

Reykjavik	Oslo	Paris	Madrid	Sydney
-2°C	___°C	6°C	___°C	24°C
___°F	35.6°F	___°F	53.6°F	___°F

Exercise 13

The charges of two car-hire companies are:

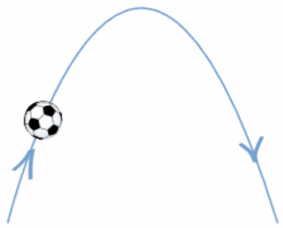
Company 1: 50 € hire fee and 0.2 € per covered kilometre.

Company 2: 20 € hire fee and 0.3 € per covered kilometre.

a) Express the charge of the companies as functions of the covered kilometres.
b) Draw both graphs in the same x and y-axes. Use them to find which company is the most advantageous for the costumers.

8.5.- PARABOLAS AND QUADRATIC FUNCTIONS

If you kick a soccer ball (or shoot an arrow, fire a missile or throw a stone) it will arc up into the air and come down again following the path of a **parabola**.



A function whose graph is a parabola is called a **quadratic function**.

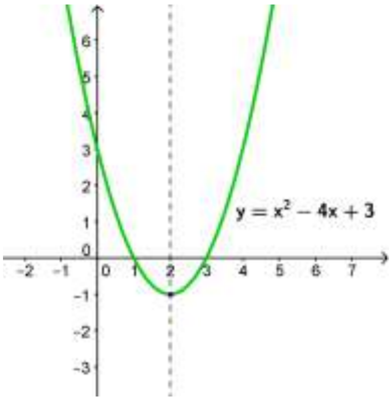
The equation of a **quadratic function** is

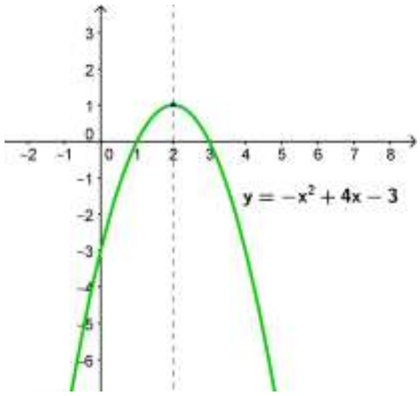
$$y = ax^2 + bx + c \quad \text{, where } a \neq 0.$$

Examples: $y = x^2$, $y = x^2 - 4$, $y = -3x^2 + 2x + 1$

The domain of a quadratic function is $\mathbb{R}$.

The parabola will open **upward** or **downward**.





A parabola will have either an absolute minimum or an absolute maximum. This point is called the **vertex** of the parabola.

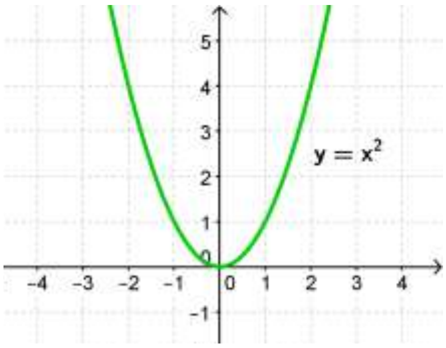
There is a line of symmetry which will divide the graph into two halves. This line is called the **axis of symmetry** of the parabola.

The parabola $y = x^2$

The basic parabola is $y = x^2$.

The function is symmetrical about the x-axis. Its vertex is the point $(0, 0)$, which is also the absolute minimum.

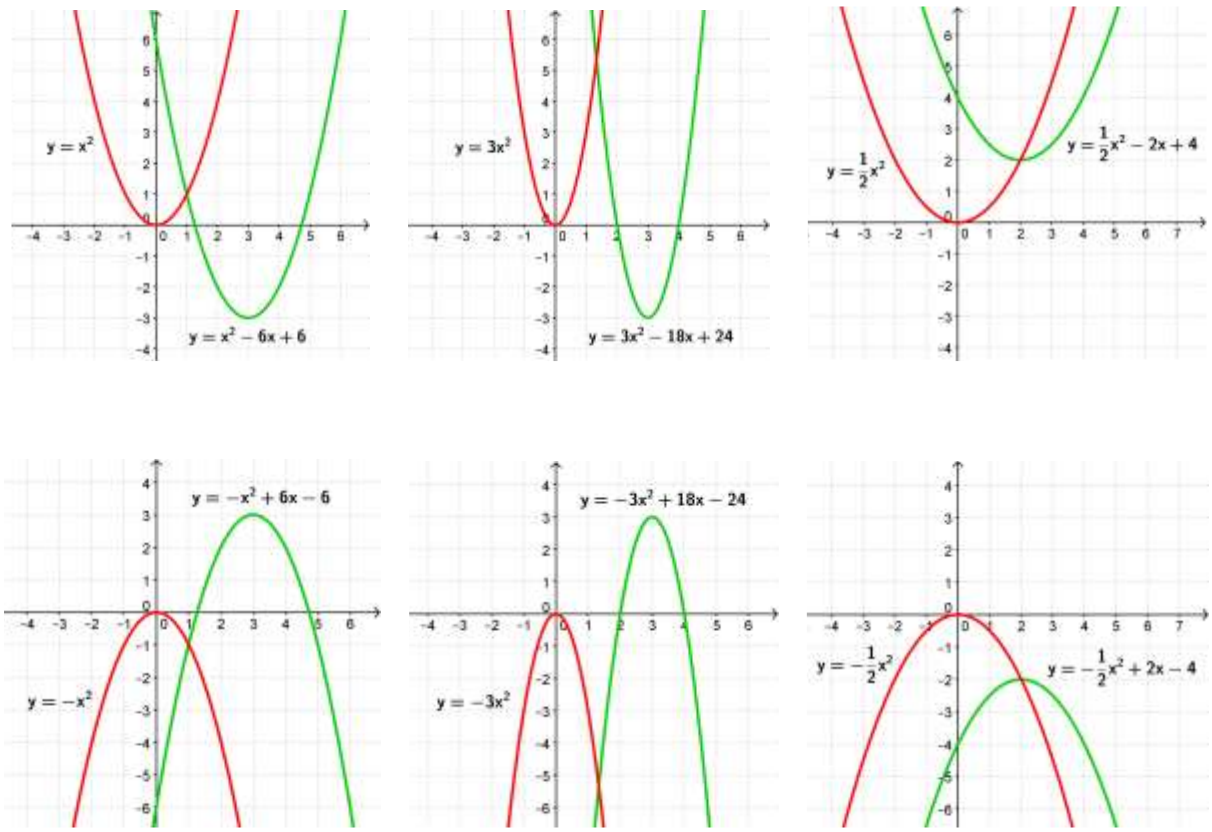
The graph has two branches (one of them is decreasing and the other one is increasing).



Graphs of the remaining quadratic functions are similar to the graph of $y = x^2$.

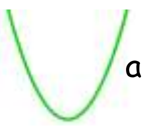
Quadratic functions

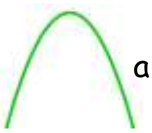
Look at these graphs:



The conclusions we can draw from these graphs are:

- The graph of a quadratic function $\mathbf{y = ax^2 + bx + c}$ is a parabola.
- Quadratic functions are continuous in $\mathbb{R}$.
- The axis of symmetry of a parabola is parallel to the y-axis.
- If two quadratic functions have the same "a", the corresponding parabolas are equal, but they are placed in different positions.
- If $a > 0$, the parabola opens upward.
  If $a < 0$, the parabola opens downward.


$a > 0$


$a < 0$

- The greater is $|a|$, the slimmer the parabola will be.

Graphing quadratic functions

A parabola $\mathbf{y = ax^2 + bx + c}$ can be represented from these points:

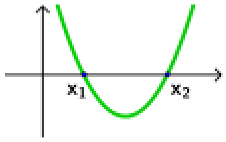
1) Axes intercept points.

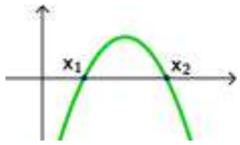
<u>x-intercepts</u>:

An x-intercept is a point on the graph where $y = 0$.

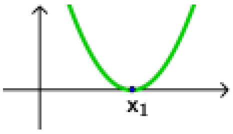
If $y = 0 \Rightarrow ax^2 + bx + c = 0$. When we solve the equation we can have:

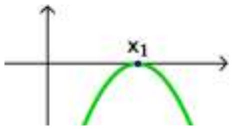
-Two different real solutions: x_1, x_2.
Then there are two x-intercept points:
$(x_1, 0)$, $(x_2, 0)$



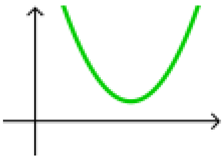


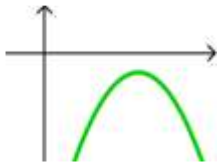
- One double real solution: $x_1 = x_2$.
Then there is only one x-intercept point:
$(x_1, 0)$





- No real solutions. Then the graph does not intercept the x-axis.

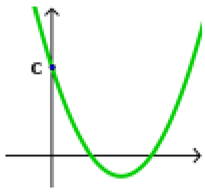




<u>y-intercept</u>:

A y-intercept is a point on the graph where $x = 0$.

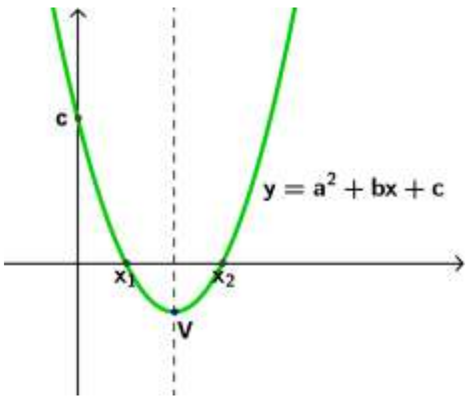
If $x = 0 \Rightarrow y = c$. Then the y-intercept point is $(c, 0)$.



2) Vertex: $V(x_v, y_v)$

$x_v = \dfrac{-b}{2a}$. The axis of symmetry of the parabola is the vertical line $x = x_v$.

3) Plot some points whose abscissa is close to the vertex of the parabola (on both sides of it).



Example: Graph the quadratic function $y = x^2 + 2x - 3$.

1) Axes intercept points.

<u>x-intercepts</u>:

$$y = 0 \Rightarrow x^2 + 2x - 3 = 0 \Rightarrow \begin{cases} x_1 = 1 & \Rightarrow (1,0) \\ x_2 = -3 & \Rightarrow (-3,0) \end{cases}$$

<u>y-intercept</u>:

If $x = 0 \Rightarrow y = -3 \Rightarrow (0,-3)$

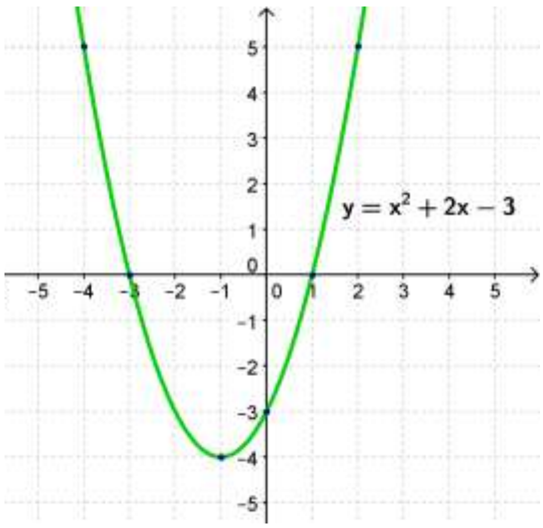
2) Vertex: $V(x_v, y_v)$

$$x_v = \frac{-b}{2a} = \frac{-2}{2 \cdot 1} = -1 \Rightarrow y_v = (-1)^2 + 2(-1) - 3 = -4 \Rightarrow V(-1,-4)$$

3) Table of values:

$y = x^2 + 2x - 3$

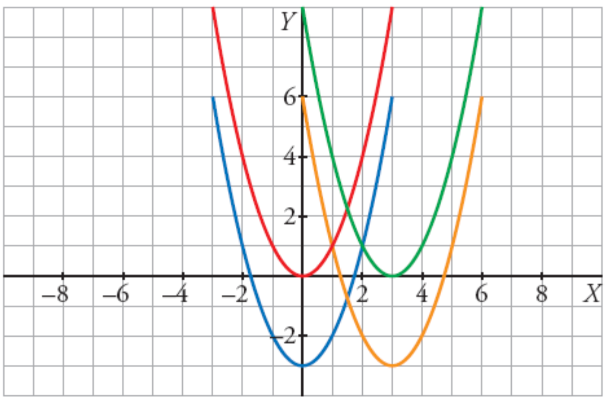
x	y
1	0
-3	0
0	-3
-1	-4
2	5
-4	5



Exercise 14

Match each parabola with its equation.

a) $y = x^2$ b) $y = x^2 - 3$ c) $y = (x-3)^2$ d) $y = x^2 - 6x + 6$



Exercise 15

Graph the following quadratic functions.

a) $y = x^2 - 5$ b) $y = x^2 - 2x + 1$ c) $y = -x^2 - 1$

d) $y = -x^2 + 5x$ e) $y = x^2 - 2x + 5$ f) $y = -2x^2 + 10x - 8$

Exercise 16

A ball is thrown into the air. The function $h = 20t - 5t^2$ shows its height, "h" metres, above the ground "t" seconds later it is thrown with an initial speed of 20 m/s.

a) Graph the function and find its domain.
b) Find the maximum height reached by the ball and the time at which it reaches this height.
c) Find the interval of time when the ball is above 15 metres?

Exercise 17

The annual cost, in euros, of producing "x" computers in a company is $C(x) = 20000 + 250x$. The annual income, in euros, that the company gets by selling "x" computers is $I(x) = 600x - 0.1x^2$.

How many computers does the company have to sell each year to maximize the profit?

Exercise 18

An Athletic field with a perimeter of 400 m consists of a rectangle with a semicircle at each end, as shown below. Find the dimensions that yield the greatest possible area for the rectangular region.