# Unit 8: PERCENTAGES. PROPORTIONALITY

## 8.1.- FRACTION, DECIMAL AND PERCENTAGE EQUIVALENTS

All these amounts show part of a whole:  $\dfrac{3}{4}$ , 0.25 , 70%

A **percentage** is a fraction written as a number of parts per 100.

For example: $60\% = \dfrac{60}{100}$  and  $132\% = \dfrac{132}{100}$

You should be able to convert between fractions, decimals and percentages …

➢ Using place value: $63\% = \dfrac{63}{100} = 0.63$    $1.43 = \dfrac{143}{100} = 143\%$

➢ Cancelling factors: $70\% = \dfrac{70}{100} = \dfrac{7}{10}$

➢ Finding an equivalent fraction: $\dfrac{1}{4} = \dfrac{25}{100} = 25\%$

➢ By dividing the numerator by the denominator of a fraction.

For example: $\dfrac{3}{4} = 3 \div 4 = 0.75$

Once you change a fraction to a decimal you can easily convert it to a percentage. You just multiply the decimal by 100.

Example: $\dfrac{3}{4} = 3 \div 4 = 0.75 = 75\%$

### *Exercise 1*

Copy and complete this table:

| Fraction (in its simplest form) | | $\dfrac{7}{16}$ | | $\dfrac{3}{11}$ | | |
|---|---|---|---|---|---|---|
| Decimal (up to 3 dp) | 0.7 | | | | 0.45 | |
| Percentage (to 1 dp) | | | 120% | | | 32.5% |

## Exercise 2
These are the marks which Rufus scored in each of his exams:

| French: $\dfrac{38}{50}$ | German: $\dfrac{17}{20}$ | History: $\dfrac{31}{45}$ |
|---|---|---|
| Maths: 77% | English: $\dfrac{41}{57}$ | Geography: 72% |

a) In which subject did he do the best? Explain your answer.
b) Put the marks for each subject in order, starting with the lowest.
c) Convert all the marks to percentages.

## 8.2.- PERCENTAGES OF AMOUNTS

Look at this example:

**Example:** There are 300 cars in a car park. 20% of the cars are red. How many cars are red?

A percentage is a fraction out of 100.

20% means 20 parts per 100 (or 20 parts in every 100) $\Rightarrow 20\% = \dfrac{20}{100}$

20% of 300 is the same as $\dfrac{20}{100}$ of 300. Remember that you find fractions of a quantity by multiplying.

$$20\% \text{ of } 300 \text{ is } \dfrac{20}{100} \cdot 300 = \dfrac{20 \cdot 300}{100} = 60$$

Therefore, there are 60 red cars in the car park.

> You can calculate the percentage of an amount using mental methods, using an equivalent fraction or using an equivalent decimal.

- Use mental methods to find simple percentages
  **Example:** 50% of 400.
  50% of a quantity is the same as one half of that quantity, so 50% of 400 are 200.

- Change the percentage to its equivalent fraction and multiply by the amount.
  **Example:** 9% of 24 m.

$$9\% \text{ of } 24 \text{ m} = \dfrac{9}{100} \times 24 = \dfrac{9 \times 24}{100} = \dfrac{216}{100} = 2.16$$

- Change the percentage to its equivalent decimal and multiply by the amount.

   ***Example:*** 37% of £58.

   37% of £58 $= \dfrac{37}{100} \times 58 = 0.37 \times 58 = £21.46$

   (0.37 is the <mark>decimal equivalent</mark> of 37%)

## *Exercise 3*

Calculate these, using a mental or written method as appropriate:
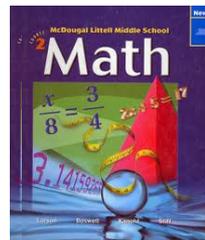
   a) 15% of £27    b) 10% of 340 m    c) 125% of 84 cm    d) 25% of 160 kg

## *Exercise 4*

A suit is designed from a material containing 14% cotton. If the suit weighs 385 grams, what is the weight of cotton in the suit?

## *Exercise 5*

A mathematics textbook has 420 pages. 35% of the pages are on number, 30% are on algebra and the remainder are on shape. How many pages are there on shape?
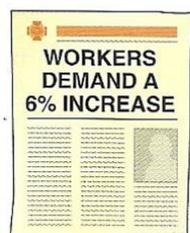
## *Exercise 6*

Three women ran for eight minutes around a track: Rachel ran $\dfrac{3}{7}$ of 1500 m, Phoebe ran 65% of 1 km and Monica ran 161% of 400 m.

   a) How far did each woman run?
   b) How far ahead of the last runner was the first runner?

## 8.3.- PERCENTAGE INCREASE AND DECREASE

In real life, lots of things are increased or reduced by a percentage.

> You can add on a percentage increase or subtract a percentage decrease at the end of the calculation.

*Example 1:* A computer costing £540 is increased in price by 15 %. What is the new price of the computer?

Original price = £540        Percentage increase = 15%

Increase = 15% of the original price = 15% of £540 = £81

New price = original price + increase = £540 + £81 = £621

*Example 2:* A book costing £18 is reduced in a sale by 22%. What is the sale price?

Original price = £18        Percentage decrease = 22%

Reduction = 22% of the original price = 22% of £18 = £3.96

Sale price = original price - reduction = £18 - £3.96 = £14.04

Alternatively you can calculate the percentage increase or decrease in a single calculation.

*Example 1:* A computer costing £540 is increased in price by 15 %. What is the new price of the computer?

New price = (100 + 15)% of original = 115% of £540 = 1.15 · £540 = £621



*Example 2:* A book costing £18 is reduced in a sale by 22%. What is the sale price?

Sale price = (100 - 22)% of original = 78% of £18 = 0.78 · £18 = £14.04



## Exercise 7

Gill used to weigh 72 kg. After going on a diet she has reduced her weight by 24%. What is her new weight?

## Exercise 8

A meal costs £28 plus a 12% service charge. How much is the bill?

### Exercise 9

Dermott buys 26 paving stones at £3.80 each and 3 bags of cement at £5.25 a bag. He is charged VAT at 17.5% on all items. What is his final bill (to the nearest penny)?

### Exercise 10

A shirt is originally priced at £19, but during a sale it is reduced by 35%. How much does the shirt cost now?

### Exercise 11

The population of Manchester in 1881 was about 1 148 000. During the next 120 years the population increased by 220%. What was the population of Manchester in 2001?

### Exercise 12

In February, a camera is on sale in Photo World for £289. In March, the manager reduces the price by 15%. In April, she reduces the new price by a further 15%.

She makes a banner saying that the camera has been reduced by 30% since February.

Is the banner telling the truth? Explain and justify your answer.

### Exercise 13

Bernard invests £3000 in a savings account. Each year the total value of the money increases by 6%.

- a) How much money is there in the account at the end of the first year?
- b) How much money is there in the account after 2 years?
- c) How many years will it take for the money to double in value?

## 8.4.- PROPORTION

A **proportion** compares the size of the part (or portion) to the size of the whole.

In a class of 30, 18 are girls and 12 are boys.

The proportion of girls is 18 out of 30.
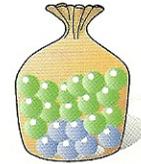The proportion of boys is 12 out of 30.

You can express a proportion as a fraction, a decimal or a percentage.

$$\text{Proportion of girls} = \frac{18}{30} = \frac{3}{5} = 0.6 = 60\%$$

$$\text{Proportion of boys} = \frac{12}{30} = \frac{2}{5} = 0.4 = 40\%$$

## Exercise 14

In a bag of 20 marbles, 14 are green and the rest are blue.
What proportion are green?

## Exercise 15

Mylene asked 120 pupils in Year 8 which subject they liked best.

| Subject | Boys | Girls |
|---|---|---|
| Science | 10 | 14 |
| English | 7 | 15 |
| Maths | 13 | 13 |
| Art | 2 | 8 |
| PE | 10 | 2 |
| Languages | 1 | 9 |
| History | 3 | 7 |
| Geography | 4 | 2 |
| Total | 50 | 70 |

a) Which subject was chosen by 8% of boys?
b) What proportion of the girls chose Languages?
c) Ilyas said that Maths was equally popular with boys and girls. Catherine said that Science was equally popular with boys and girls. Are they right or wrong?

## 8.5.- INTRODUCING RATIO

**Proportion** compares the size of a part with the size of the whole.
**Ratio** compares the size of two portions of parts with each other

On this stick, 6 cm is painted red and 9 cm is painted blue.

| | |
|---|---|
| The proportion of the stick that is blue<br>$= 9$ cm out of 15 cm<br>$= \frac{9}{15} = \frac{3}{5} = 60\%$<br>3 cm in every 5 cm is blue | The ratio of red to blue<br>$= 6$ cm to 9 cm<br>$= 6:9$ (common factor is 3)<br>$= 2:3$<br>For each 2 cm of red there is 3 cm of blue |
| The proportion of the stick that is red<br>$= 6$ cm out of 15 cm<br>$= \frac{6}{15} = \frac{2}{5} = 40\%$<br>2 cm in every 5 cm is red | The ratio of blue to red<br>$= 9$ cm to 6 cm<br>$= 9:6$ (common factor is 3)<br>$= 3:2$<br>For each 3 cm of blue there is 2 cm of red |

You use a ratio to compare the sizes of any two quantities.
Ratios, like fractions, can be cancelled down by dividing, for example:
$$6:15 = 2:5 \text{ (common factor is 3).}$$

You can split quantities into different proportion using a ratio.

*Example:* Pinky and Perky share £200 in the ratio $3:7$ . How much will each receive?

For every £2 Pinky gets, Perky gets £7.
The total number of parts is $3+7=10$.

The proportion that Pinky will get is $\dfrac{3}{10}$ and Perky will get $\dfrac{7}{10}$.

$\dfrac{3}{10}$ of £200 $= 3 \cdot \dfrac{1}{10}$ of £200 $= 3 \cdot £20 = £60$

$\dfrac{7}{10}$ of £200 $= 7 \cdot \dfrac{1}{10}$ of £200 $= 7 \cdot £20 = £140$

Pinky will receive £60 and Perky £140.
(check: $60 + 140 = 200$ )

> £200 will be shared into 10 parts so each part is £200 : 10 = £20.
> Pinky has three parts $= 3 \cdot £20 = £60$.
> Perky has seven parts $= 7 \cdot £20 = £140$.

### Exercise 16

Write these ratios in their simplest form:

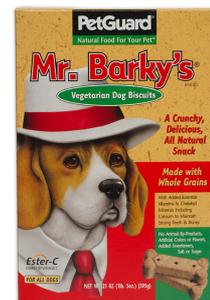    a) $9:15$      b) $7:21$      c) $15:20$      d) $15:85$

### Exercise 17

There are 2 men and for every 3 women at the gym club. There are 15 women at the club. How many men are there?

### Exercise 18

44 dog biscuits are shared between three dogs, Toby, Tess and Molly, in the ratio of their ages. In dog years, Toby is 70, Tess is 56 and Molly is 28. How many dog biscuits does each dog receive?

## 8.6.- DIRECT PROPORTION

Suppose the price of a can of soup is £0.50. If a person wants to buy 12 cans of soup, then he has to pay £6. If he wants to buy 24 cans of soup, he has to pay £12 and so on.

We can easily see that if the person buys more cans, he has to pay more or he has to pay less if he buys less cans.

| Number of cans | Total price |
|----------------|-------------|
| 12             | 6           |
| 24             | 12          |
| 36             | 18          |

That is, as number of cans is increased total price also increased, conversely, if the number of cans is decreased total price also decreased. In such situation, we say that number of cans and price are directly related.

> When an increase in one quantity means another quantity increases in the same proportion, the quantities are in **direct proportion**.

If x and y are in direct proportion, then the division of x and y will be constant.

$$\frac{x}{y} = constant$$

In the above example, we see that:

$$\frac{12}{6} = 2 \qquad \frac{24}{12} = 2 \qquad \frac{36}{18} = 2 \qquad \Rightarrow \quad \text{Each ratio is the same.}$$

Hence, if we are dealing with quantities which are in direct proportion, then we can use the follow rule:

<u>Number of cans</u>                     <u>Total price</u>
       12       --------------       6
       36       --------------       18

$$\frac{12}{36} = \frac{6}{18} \quad \Rightarrow \quad 12 \cdot 18 = 36 \cdot 6$$

In general:

> <u>Quantity 1</u>                  <u>Quantity 2</u>
>     a       ------------       c
>     b       ------------       d
>
> $$\frac{a}{b} = \frac{c}{d} \quad \Leftrightarrow \quad a \cdot d = b \cdot c$$

***Example:*** With 8 gallons of petrol my car travels 248 miles. How far will it travel with 11 gallons of petrol?

$$\begin{array}{ccc} \underline{\text{Gallons}} & & \underline{\text{Miles}} \\ 8 & \text{------------} & 248 \\ 11 & \text{------------} & x \end{array}$$

$$\frac{8}{11} = \frac{248}{x} \Rightarrow 8 \cdot x = 11 \cdot 248 \Rightarrow x = \frac{11 \cdot 248}{8} = 341 \text{ miles}$$

## Exercise 19

A distance of 8 km is represented by 15.2 cm on a map. How many cm will represent 10 km?

## Exercise 20

A family spends 120 dollars per month for vegetables when the cost of vegetables is 5 dollars per kg. How much amount must the family spend per month, if the cost of vegetables goes up to 5.75 dollars per kg?

## Exercise 21

One evening the length of the shadow of a building's pillar of height 28 m is 42 m. What will be the length of the shadow of the building's tower of height 36 m at that time?

## Exercise 22

The weight E of a man on earth's surface is directly proportional to his weight M on mars. The constant of proportionality is 3, if a man weights 126 kg on earth, how much will he weigh on mars?

## Exercise 23

The numbers p and q are directly proportional and $p = 7.5$ when $q = 15$. Find p when $q = 38$; and find q when $p = 13$.

## Exercise 24

A store sells packs of paper in two sizes.

| Regular | Super |
|---|---|
| 150 sheets | 500 sheets |
| Cost £1.05 | Cost £3.85 |

Which of these two packs give better value for money? You must show all of your working.

## 8.7.- INVERSE PROPORTION

Suppose that 20 men build a house in 6 days. If men are increased to 30 then take 4 days to build the same house. If men become 40, they take 3 days to build the house.

We can easily see that if the number of men is increased, the time taken to build the house is decreased in the same proportion.

| Number of men | Number of days |
|:---:|:---:|
| 20 | 6 |
| 30 | 4 |
| 40 | 3 |

When an increase in one quantity means another quantity decreases in the same proportion, the quantities are in **inverse proportion**.

If x and y are in inverse proportion,
then the product of x and y will be constant.

$$x \cdot y = constant$$

In the above example, we see that:

$$20 \cdot 6 = 120 \qquad 30 \cdot 4 = 120 \qquad 40 \cdot 3 = 120$$

If we are dealing with quantities which are in inverse proportion, then we can use the follow rule:

| Number of men | | Number of days |
|:---:|:---:|:---:|
| 20 | ------------- | 6 |
| 30 | ------------- | 4 |

Notice that $\dfrac{4}{6}$ is the multiplicative inverse of $\dfrac{6}{4}$.

$$\frac{20}{30} = \boxed{\frac{4}{6}} \Rightarrow 20 \cdot 6 = 30 \cdot 4$$

In general:

| Quantity 1 | | Quantity 2 |
|:---:|:---:|:---:|
| a | ------------ | c |
| b | ------------ | d |

$$\frac{a}{b} = \frac{d}{c} \Leftrightarrow a \cdot c = b \cdot d$$

***Example:*** The groceries in a home of 4 members are enough for 30 days. If a guest comes and stays with them, how many days will the groceries last?

$$
\begin{array}{ccc}
\underline{\text{Members}} & & \underline{\text{Days}} \\
4 & \text{------------} & 30 \\
5 & \text{------------} & x
\end{array}
$$

$$\frac{4}{5} = \frac{x}{30} \implies 4 \cdot 30 = 5 \cdot x \implies x = \frac{4 \cdot 30}{5} = 24 \text{ days}$$

## Exercise 25

At the rate of 28 lines per page, a book has 300 pages. If the book has to contain only 280 pages, how many lines should a page contain?

## Exercise 26

Maria cycles to her school at an average speed of 12 km/h. It takes her 20 minutes to reach the school. If she wants to reach her school in 15 minutes, what should be her average speed?

## Exercise 27

Some people working at the rate of 6 hours a day can complete the work in 19.5 days. As they have received another contract, they want to finish this work early. Now they start working 6.5 hours a day. In how many days will they finish this work?

## Exercise 28

The numbers $x$ and $y$ are inversely proportional and $x = 6$ when $y = -3$. Find $x$ when $y = -9$; and find $y$ when $x = -4.5$.

## Exercise 29

In an army camp, there is food for 5 weeks for 1200 people. If 300 more soldiers joined the camp, for how many weeks will the food last?