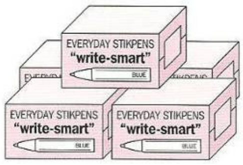
Unit 9: ALGEBRA

9.1.- ALGEBRAIC EXPRESSIONS

You can describe everyday situations using algebra.

In **algebra**, you use letters to represent unknown numbers.

These boxes hold n pens each.



In 5 boxes there are $n+n+n+n+n=5\times n=5n$ pens

These boxes hold s pens each

In 3 boxes there are $3s$ pens

There are $5n+3s$ pens in total.

$5n+3s$ is an **algebraic expression**.

An algebraic expression has numbers and letters linked by operations. The letters are called **variables**. Every addend is called **term**.

The expression $5n+3s$ has two terms: **5n** and **3s**.

You can simplify an algebraic expression by collecting **like terms**.

Like terms have exactly the same letters. ($3x^2$ and $-5x^2$ are like terms).

Example: Simplify these expressions:

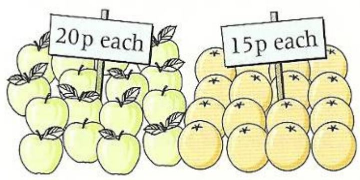
a) $4x+2y-2x+3y$ b) $7p-3q+5q-p$

a) $4x+2y-2x+3y=4x-2x+2y+3y=2x+5y$

b) $7p-3q+5q-p=7p-p+5q-3q=6p+2q$

Example:

In a fruit shop, apples cost 20p each and orange cost 15p each. Write an expression for the cost of x apples and y oranges.



Cost of x apples: $20x$ Cost of oranges: $15y$

Total cost: $20x+15y$

Exercise 1

In one month, Dan sends x texts.

a) Alice sends 4 times as many texts as Dan. How many is this?
b) Kris sends 8 more texts than Alice. How many is this?

Exercise 2

In a pizza takeaway

- a medium pizza has 6 slices of tomato
- a large pizza has 10 slices of tomato

How many slices of tomato are needed for c medium pizzas and d large pizzas?

Evaluating algebraic expressions

You can find the value of an algebraic expression when you know the value of the letters used.

Example: Find the value of $20x + 15y$ when $x = 5$ and $y = 8$.

We substitute the values of x and y in the expression:

$$20 \cdot 5 + 15 \cdot 8 = 100 + 120 = 220$$

Evaluating an algebraic expression is the same as calculating its number value at a given value of the variables.

Exercise 3

Find the value of these algebraic expressions when $p = 3$, $q = 5$ and $r = 6$.

a) $\frac{4p + r}{2}$ e) $6 - 2r$ i) $r + pq$ n) pqr

o) $2(3q + 1)$ q) $pq - 3r$ t) $4 + 3p$ u) $pr - 2q$

Rearrange yours answers in order, smallest to largest. What mathematical word do the letters spell?

Exercise 4

Work out the value of these expressions when $p = 5$.

a) p^3 b) $3p^2$ c) $(3 + p)^2$ d) $3 + p^2$ e) $\frac{2p^2}{10}$

Exercise 5

2n and n^2 are different expressions.

a) Find a value for n to show that the two expressions have different values.
b) Find a value for n that gives the same value for each of the expressions.
c) Find a value for n where 2n is larger than n^2.

9.2.- MONOMIALS

A **monomial** is an algebraic expression containing one term which may be a number, a variable or a product of numbers and variables, with no negative or fractional exponents. (*Mono* implies *one* and the ending *nomial* is Greek for *part*)

For example: $\frac{2}{3}a^2b$, x^2 , $-2xy$, 13 , $520x^2y^4$ are monomials

x^{-1} , $x^{\frac{1}{2}}$ are NOT monomials

The number is called **coefficient** and the variables are called **literal part**.

$$\frac{2}{3}a^2b$$

coefficient   literal part

The **degree** is the sum of the exponents of every variable.

Example: the degree of $\frac{2}{3}a^2b$ is $2+1=3$

Exercise 6

For every monomial write the coefficient, the literal part and the degree:

Monomial	$5x$	$0.3x^4$	$-\frac{1}{9}x^2y$	$\frac{1}{8}xy$	$-\frac{3}{7}x^2y^3$	$-x^2y^4$
Coefficient						
Literal part						
Degree						

Adding and subtracting monomials

You can add or subtract monomials only if they have the same literal part, that is, if they are **like terms**. In this case, you sum or subtract the coefficients and leave the same literal part.

Examples: $4xy^2 + 3xy^2 = 7xy^2$ $5x^2 + 3 - 2x^2 - 1 = 3x^2 + 2$

Exercise 7

Simplify these algebraic expressions.

a) $a + a + a + a$ b) $m + m - m$ c) $x + x + y + y + y$ d) $4a + 3a$
e) $9x - 5x$ f) $6a + 2a - 5a$ g) $10x - 3x - x$ h) $8a^2 - 3a^2 - a^2$

Exercise 8

Remove brackets and simplify.

a) $5x^2 - (2x + x^2)$ b) $3x - (x - x^2)$ c) $x^2 - (3x - x^2)$
d) $(x^2 + x) + (3x + 1)$ e) $(5x^2 - 4x) - (2x^2 + 2x)$ f) $(4x^2 - 5x) - (2x^2 + 2)$

Exercise 9

At a pick-your-own fruit farm, Lucy picks n apples. Mary picks 5 more apples than Lucy. Nat picks 3 times as many apples as Mary.

a) Write in terms of n, the number of apples Mary picks.
b) Write in terms of n, the number of apples Nat picks.

Exercise 10

Jake is *n* years old.
Jake's sister is 4 years older than Jake.
Jake's mother is 3 times older than his sister.
Jake's father is 4 times older than Jake.
Jake's uncle is 2 years younger than Jake's father.
Jake's grandmother is twice as old as Jake's uncle.

a) Copy the table and write each person's age in terms of *n*.

Jake 	Sister 	Mother 	Father 	Uncle 	Grandmother 
n					

b) Find, in terms of *n*, how much older Jake's grandmother is than his mother. Give your answer in its simplest form.

Multiplying monomials

The product of two monomials is always another monomial.

Examples:

- $(2x)\cdot(4y)=2\cdot x\cdot 4\cdot y=2\cdot 4\cdot x\cdot y=8xy$
- $(-2a)\cdot(5a)=(-2)\cdot a\cdot 5\cdot a=(-2)\cdot 5\cdot a\cdot a=-10a^2$
- $\left(\frac{1}{3}x\right)\cdot(6xy)=\frac{1}{3}\cdot x\cdot 6\cdot x\cdot y=\frac{1}{3}\cdot 6\cdot x\cdot x\cdot y=\frac{6}{3}x^2y=2x^2y$

If one of the factors is a sum of monomials, distribute the monomial through the brackets.

Examples:

- $3x(4x^2+10)=3x\cdot 4x^2+3x\cdot 10=12x^3+30x$
- $-2x(3x^4-5y)=-6x^5+10xy$

Multiply all the terms inside the bracket by the term outside is called **expanding the bracket**.

Dividing monomials

When you divide two monomials you can get a number, a monomial or an algebraic fraction.

Examples:

- $(2a):(8a)=\frac{2a}{8a}=\frac{2}{8}\cdot\frac{a}{a}=\frac{1}{4}\cdot 1=\frac{1}{4}$
- $(12x^2y):(-4xy)=\frac{12x^2y}{-4xy}=\frac{12}{-4}\cdot\frac{x^2}{x}\cdot\frac{y}{y}=-3x$
- $(-10xy):(-2y^2)=\frac{-10xy}{-2y^2}=\frac{-10\cdot x\cdot \not{y}}{-2\cdot y\cdot \not{y}}=\frac{5x}{y}$

Exercise 11

Multiply the following monomials.

a) $x\cdot 2x$ b) $a^3\cdot a^2$ c) $3a\cdot 4a^2$ d) $(2a)\cdot(-4ab)$

e) $(-xy^2)\cdot(3x^2y)$ f) $(5a^2)\cdot(2ab)$ g) $(3a^2b^3)\cdot(a^2b)$ h) $\left(\frac{3}{5}x^2\right)\cdot\left(\frac{2}{9}x^4\right)$

Exercise 12

Divide the following monomials.

a) $x^2 : x$ b) $a^5 : a^2$ c) $b^4 : b^4$ d) $x^2 : x^3$

e) $m^2 : m^5$ f) $10x^4 : (-5x)$ g) $6a^2 : 9a^5$ h) $12x^2 : (-4x^2)$

Exercise 13

Simplify these algebraic fractions.

a) $\dfrac{10x}{5x^3}$ b) $\dfrac{3ab}{9a^2}$ c) $\dfrac{4a^2b}{8ab^2}$ d) $\dfrac{2ab}{10a^2b^2}$

9.3.- IDENTITIES, FORMULAE AND EQUATIONS

An **identity** is true for all values of x. For example:

$x(x+1) \equiv x^2 + x$ Whatever value of x you try, this statement is always true.

$\equiv$ means 'is identical to'

An **equation** is only true for a limited number of values of x. For example:

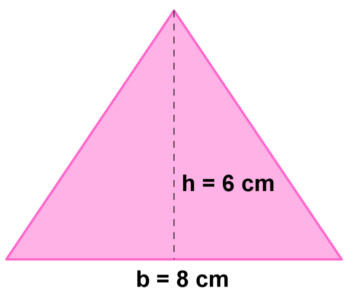
$2x+1=5$ is only true when $x=2$.

A **formula** describes the relationship between two or more variables.

For example: the formula for the area of a triangle is $A = \frac{1}{2}bh$

You can **substitute** numbers into a formula to work out the value of a variable.

For example:



$$A = \frac{1}{2}bh = \frac{1}{2} \times 8 \times 6 = 24 \ \text{cm}^2$$

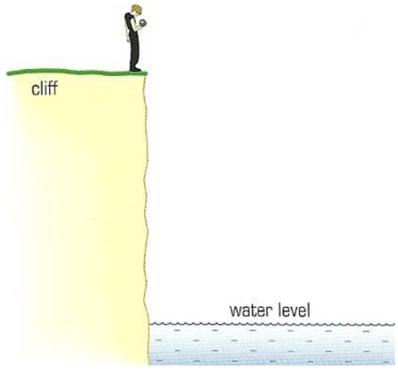
Exercise 14

Decide if each of these statements is an identity, an equation or a formula.

a) $x^3 - 2x = x(x^2 - 2)$ b) $3x - 1 = 17$ c) $A = \pi r^2$

Exercise 15

A boy drops a stone over the edge of a cliff. The stone takes 3 seconds before it splashes into the sea at the bottom of the cliff.



a) Use the formula $v = at$ to find the velocity (speed) of the stone as it splashes into the sea.

b) t seconds later the stone is dropped, its height above the sea level is given by the formula $h = \frac{1}{2}at^2$. How high was the cliff above sea level?

v = velocity (speed) in m/s
a = acceleration = 10 m/s^2
t = time in seconds = 3 seconds
h = height (in metres)

9.4.- FIRST-DEGREE EQUATIONS

A **first-degree equation** is called a **linear equation**. The highest exponent of a linear equation is 1. The standard form for a linear equation is:

$$ax + b = c$$

a, **b** and **c** can have any value, except that **a** can't be 0

Examples: $x + 5 = 8$, $3x = 15$, $2x + 1 = 9$

The **solution** of an equation is the set of values which, when substituted for unknowns, make the equation a true statement.

$x + 5 = 8 \Rightarrow \underline{\underline{x = 3}}$ $3x = 15 \Rightarrow \underline{\underline{x = 5}}$ $2x + 1 = 9 \Rightarrow \underline{\underline{x = 4}}$

Solving basic equations

In an equation the equals sign shows that the sides balance. To solve an equation you must always keep the balance.

Example 1:

$$x + 3 = -2$$
$$x + 3 - 3 = -2 - 3$$
$$x = -2 - 3$$
$$x = -5$$

Tip: we "leave out" the first step and we say: terms can "move" from the left-hand side to the right-hand side (and vice versa) by changing their sign.

$$x + 3 = -2$$
$$x = -2 - 3$$
$$x = -5$$

Example 2:

$$3x = 15$$
$$\frac{3x}{3} = \frac{15}{3}$$
$$x = 5$$

Tip: we "leave out" the first step and we say: the number that is multiplying on one side "is moved" to the other side and its terms are divided by it (and vice versa).

$$3x = 15$$
$$x = \frac{15}{3}$$
$$x = 5$$

Example 3:

$$6x - 7 = 3x + 5$$
$$6x - 3x = 5 + 7$$
$$3x = 12$$
$$x = \frac{12}{3}$$
$$x = 4$$

You want all the x terms on one side of the equation and all the number terms on the other side.

Example 4:

$$7 - (1 - 3x) = 12$$
$$7 - 1 + 3x = 12$$
$$3x = 12 - 7 + 1$$
$$3x = 6$$
$$x = \frac{6}{3}$$
$$x = 2$$

Exercise 16

Solve each of these equations:

a) $5x - 4x = 9$

b) $3x + 6 = 2x + 13$

c) $(2x - 1) - 1 = 5(3x - 2) + 3$

d) $5(x - 1) - 3(3x - 2) = 2$

e) $\frac{5x}{2} = 10$

f) $\frac{x}{2} + \frac{5x}{2} = 9$

g) $\frac{9}{2x} = 3$

h) $\frac{x + 4}{x} = 3$

Exercise 17

Odd one out

Which equation has a different solution to the other two?

$3x + 5 = x + 19$

$\frac{7x + 2}{3} = \frac{2x + 13}{5}$

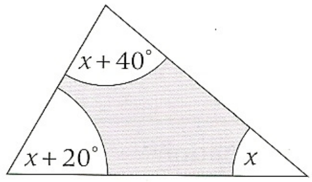
$5(x - 2) = 3x + 4$

9.5.- SOLVING PROBLEMS USING EQUATIONS

Word problems are a series of expressions that fits into an equation.

For example:

The angles in a triangle are x, x + 20 and x + 40 . Find the angles of the triangle.

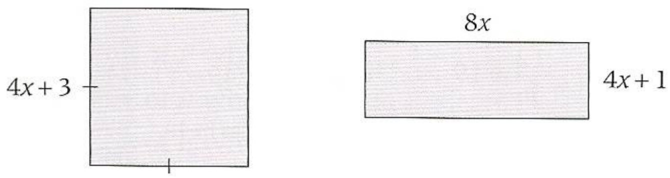


$$x + (x + 20) + (x + 40) = 180$$
$$x + x + 20 + x + 40 = 180$$
$$x + x + x = 180 - 20 - 40$$
$$3x = 120$$
$$x = \frac{120}{3}$$
$$x = 40$$

Solution: The angles of the triangle are 40°, 60° and 100°.

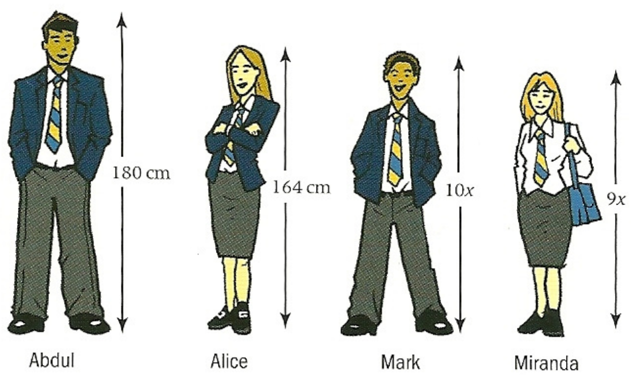
Exercise 18

The perimeters of these two shapes are equal. What are the dimensions of each shape?



Exercise 19

Abdul is 180 cm tall and Mark is 10x cm tall. Alice is 164 cm tall and Miranda is 9x cm tall. The difference in height between the two boys is equal to the difference in height between the two girls. How tall are Mark and Miranda?



Exercise 20

You have $60 and your sister has $120. You are saving $7 per week and your sister is saving $5 per week. How long will it be before you and your sister have the same amount of money?

Exercise 21

Karen is twice as old as Lori. In three years' time, the sum of their ages will be 42. How old is Karen?

Exercise 22

What are two consecutive integers, such that seven times the larger minus three times the smaller is 95?

Exercise 23

Christina and Julia were running for mayor in a small town. Christina received thirty percent of the votes. Julia received four thousand, sixty votes. How many votes were cast in the town, assuming that everybody in the town voted for either Christina or Julia?

Exercise 24

In each case, use the information to write an equation and solve it to find the starting number.

a) I think of a number, multiply it by 8 and subtract 2. I get the same answer as when I multiply this number by 2 and add 10.
b) I think of a number, multiply it by 5 and add 3. I get the same answer as when I multiply it by 2 and subtract it from 24.
c) Taking double a number from 11 is equal to taking treble that number from 14.

Exercise 25

If the larger of two numbers were decreased by three hundred forty-nine, then the two numbers would be the same. The sum of the two numbers is 735. What are the numbers?

Exercise 26

Kyle lives twenty-nine miles from work. During the summer, he rides his bike at six mph to the metro station. He then takes the metro to work, which travels at thirty-three mph. If he spends six minutes less on the metro than on his bike, how long does she take to go to work?